Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section One:
Calculator-free

SOLUTIONS

Student number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

Let $u = 1 + i$, $v = 1 + \sqrt{3}i$ and $z = u^3 v^2$.

(a) Determine the modulus and argument of z. (4 marks)

Solution
$u = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$ and $v = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$ $z = 2\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right) \times 4\operatorname{cis}\left(\frac{\pi}{3}\right)$ $= 8\sqrt{2}\operatorname{cis}\left(-\frac{7\pi}{12}\right)$ $\lvert z\rvert = 8\sqrt{2}$ and $\arg(z) = -\frac{7\pi}{12}$
Specific behaviours
✓ expresses u and v in polar form   ✓ indicates powers of u and v ✓ forms product   ✓ states modulus and argument

(b) Determine the smallest positive integer k such that z^k is real. (2 marks)

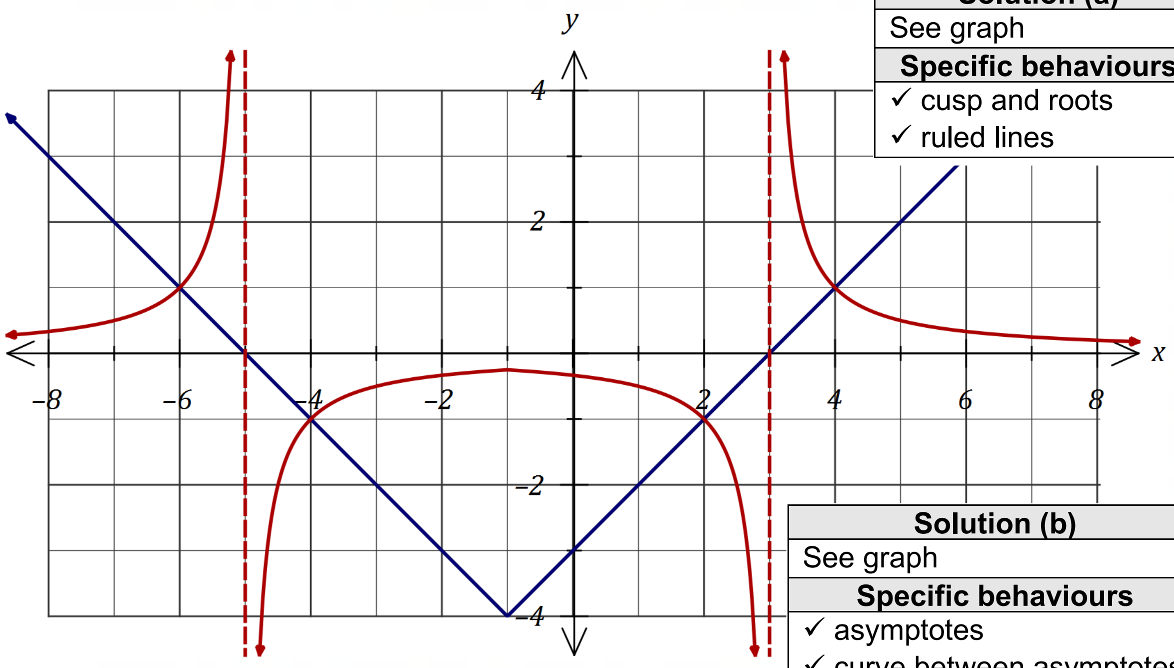
Solution
If z^k is real, then $\arg(z^k) = -\frac{7k\pi}{12}$ must be a multiple of π. Hence $k = 12$.
Specific behaviours
✓ indicates argument restriction   ✓ correct value of k

Question 2 **(6 marks)**

Let $f(x) = |x+1| - 4$.

(a) Sketch the graph of $y = f(x)$ on the axes below. (2 marks)

Solution (a)
See graph
Specific behaviours
✓ cusp and roots
✓ ruled lines



Solution (b)
See graph
Specific behaviours
✓ asymptotes
✓ curve between asymptotes
✓ curves outside asymptotes

(b) On the same axes, sketch the graph of $y = \dfrac{1}{f(x)}$. (3 marks)

(c) Determine all solutions to the equation $\left(f(x)\right)^2 = 1$. (1 mark)

Solution
$\left(f(x)\right)^2 = 1 \Rightarrow f(x) = \pm 1 \text{ or } f(x) = \dfrac{1}{f(x)}$ $x = -6, -4, 2, 4$
Specific behaviours
✓ all solutions

Question 3 **(6 marks)**

Functions f, g and h are defined as $f(x) = 1 + \sqrt{x}$, $g(x) = 4 - \ln x$ and $h(x) = f \circ g(x)$.

(a) Determine the defining rule for $g^{-1}(x)$ and its domain. (2 marks)

Solution
$y = 4 - \ln x \Rightarrow x = e^{4-y}$ $g^{-1}(x) = e^{4-x}$ $D_{g^{-1}}: \{x: x \in \mathbb{R}\}$
Specific behaviours
✓ defining rule ✓ domain

(b) Determine an expression for $h(x)$ and its domain and range. (4 marks)

Solution
$h(x) = f \circ g(x) = 1 + \sqrt{4 - \ln x}$ $\ln x \Rightarrow x > 0$ $4 - \ln x \geq 0 \Rightarrow x \leq e^4$ $D_h: \{x: 0 < x \leq e^4\}$ $R_h: \{y: y \geq 1\}$
Specific behaviours
✓ expression for $h(x)$ ✓ lower bound, inequality for domain ✓ upper bound, inequality for domain ✓ range

Question 4 **(6 marks)**

Let $g(z) = z^4 + 9z^2 + 14$ where $z \in \mathbb{C}$.

(a) Clearly show that $(z - \sqrt{7}i)$ is a factor of $g(z)$. (2 marks)

Solution
$g(\sqrt{7}) = (\sqrt{7}i)^4 + 9(\sqrt{7}i)^2 + 14$ $= 49 + 9(-7) + 14$ $= 63 - 63$ $= 0$
Specific behaviours
✓ substitutes $z = \sqrt{7}$ correctly   ✓ clearly shows terms sum to 0

(b) Solve the equation $g(z) = 0$. (4 marks)

Solution
$g(z) = (z - \sqrt{7}i)(z + \sqrt{7}i)(az^2 + bz + c)$ $= (z^2 + 7)(z^2 + 2)$ $g(z) = 0 \Rightarrow z = \pm\sqrt{7}i, z = \pm\sqrt{2}i$
Specific behaviours
✓ indicates use of conjugate for second factor   ✓ factorises $g(z)$ ✓ obtains third solution   ✓ lists all four solutions

Question 5 **(7 marks)**

A particle leaves the origin at time $t = 0$ with initial velocity $v = 4$ and moves in a straight line with acceleration given by

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x - 4$$

where v is its velocity and x is its displacement from the origin at time t, $t \geq 0$.

(a) Determine an equation for v as a function of x. (4 marks)

Solution
$\frac{1}{2}v^2 = \int x - 4\,dx$ $= \frac{1}{2}x^2 - 4x + c$ $v = 4, x = 0 \Rightarrow 8 = c$ $\frac{1}{2}v^2 = \frac{1}{2}x^2 - 4x + 8$ $v^2 = x^2 - 8x + 16 = (x-4)^2$ $v = \pm(x-4)$ $v = 4, x = 0 \Rightarrow v = -(x-4) = 4 - x$
Specific behaviours
✓ integrates both sides   ✓ determines constant of integration   ✓ simplifies and factorises   ✓ clearly considers $\pm$ root

(b) Determine an equation for x as a function of time t. (3 marks)

Solution
$\frac{dx}{dt} = 4 - x \Rightarrow \int \frac{1}{4-x}dx = \int dt$ $-\ln\lvert 4-x\rvert = t + c$ $4 - x = ke^{-t}$ $t = 0, x = 0 \Rightarrow k = 4$ $4 - x = 4e^{-t}$ $x = 4 - 4e^{-t}$
Specific behaviours
✓ separates variables and integrates   ✓ eliminates logs and evaluates constant   ✓ writes expression

Question 6 **(7 marks)**

Determine

(a) $\displaystyle\int \frac{5x+7}{x^2+3x+2}\,dx.$ (3 marks)

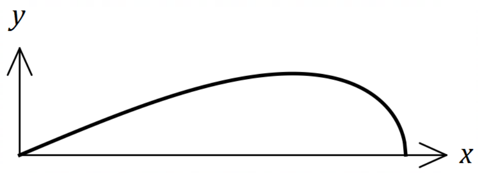
Solution
$\dfrac{5x+7}{(x+1)(x+2)} = \dfrac{2}{x+1} + \dfrac{3}{x+2}$ $\displaystyle\int \frac{2}{x+1} + \frac{3}{x+2}\,dx = 2\ln\lvert x+1\rvert + 3\ln\lvert x+2\rvert + c$
Specific behaviours
✓ indicates use of partial fractions   ✓ correct partial fractions   ✓ correct integral, including constant

(b) $\displaystyle\int \frac{1}{4+x^2}\,dx$, using the substitution $x = 2\tan\theta$. (4 marks)

Solution
$dx = 2\sec^2\theta\,d\theta$ $x^2 = 4\tan^2\theta$ $4 + x^2 = 4 + 4\tan^2\theta$ $= 4\sec^2\theta$ $\displaystyle\int \frac{1}{4+x^2}\,dx = \int \frac{2\sec^2\theta}{4\sec^2\theta}\,d\theta$ $\displaystyle= \int \frac{1}{2}\,d\theta$ $= \dfrac{\theta}{2}$ $= \dfrac{1}{2}\tan^{-1}\dfrac{x}{2} + c$
Specific behaviours
✓ relates dx and $d\theta$ ✓ uses trig identity   ✓ writes and simplifies integral wrt θ ✓ integrates and writes wrt x

Question 7 **(7 marks)**

The graph of $y = x\sqrt{16 - x^2}$ in the first quadrant is shown below.



(a) Determine the area of the region shown between the curve and the x-axis. (4 marks)

Solution
$$A = \int_0^4 x(16 - x^2)^{0.5}\,dx$$ $$u = 16 - x^2 \Rightarrow du = -2x\,dx$$ $$x = 0 \Rightarrow u = 16, x = 4 \Rightarrow u = 0$$ $$A = \int_{16}^{0} -\frac{u^{0.5}}{2}\,du$$ $$= \left[-\frac{1}{3}u^{\frac{3}{2}}\right]_{16}^{0}$$ $$= [0] - \left[-\frac{64}{3}\right] = \frac{64}{3} \text{ square units}$$
Specific behaviours
✓ correct integral in terms of x ✓ indicates valid integration method ✓ integrates correctly ✓ simplified area

(b) Determine the volume of the solid of revolution formed when the region between the curve, the x-axis, $x = 1$ and $x = 2$ is rotated about the x-axis. (3 marks)

Solution
$$V = \pi \int y^2\,dx$$ $$= \pi \int_1^2 16x^2 - x^4\,dx$$ $$= \pi \left[\frac{16x^3}{3} - \frac{x^5}{5}\right]_1^2$$ $$= \pi \left[\left(\frac{128}{3} - \frac{32}{5}\right) - \left(\frac{16}{3} - \frac{1}{5}\right)\right]$$ $$= \pi \left[\frac{112}{3} - \frac{31}{5}\right]$$ $$= \frac{467\pi}{15} \text{ cubic units}$$
Specific behaviours
✓ correct integral in terms of x ✓ integrates and substitutes correctly ✓ simplified volume

Question 8 **(7 marks)**

Three planes have equations

$$2x + 2y + 7z = 5$$
$$2x + y - 7z = 1$$
$$4x + ay = 3$$

where a is a constant.

(a) Explain why the planes cannot intersect at a unique point when $a = 3$. (2 marks)

Solution
$a = 3 \Rightarrow 4x + 3y = 3$ $\text{Eqn(1)+Eqn(2)}: 4x + 3y = 6$ Hence no solution to system as equations inconsistent and so planes cannot intersect at a point.
Specific behaviours
✓ subtracts equations ✓ explanation

The acute angle between the planes $2x + y - 7z = 1$ and $4x + ay = 3$ is θ, where $\cos\theta = \dfrac{\sqrt{6}}{18}$.

(b) Determine the value of a. (5 marks)

Solution
Let normals to planes be $\mathbf{n}_1$ and $\mathbf{n}_2$: $\mathbf{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 4 \\ a \\ 0 \end{pmatrix}$ $\lvert\mathbf{n}_1\rvert = \sqrt{54} = 3\sqrt{6}, \lvert\mathbf{n}_1\rvert = \sqrt{a^2 + 16}$ $\mathbf{n}_1 \cdot \mathbf{n}_1 = a + 8$ $a + 8 = 3\sqrt{6} \times \sqrt{a^2 + 16} \times \dfrac{\sqrt{6}}{18}$ $a + 8 = \sqrt{a^2 + 16}$ $a^2 + 16a + 64 = a^2 + 16$ $16a = -48$ $a = -3$
Specific behaviours
✓ magnitudes of normals ✓ dot product of normals ✓ equation using scalar product ✓ squares both sides ✓ correct value of a

End of questions

Supplementary page

Question number: _________